Exam 2021

P1 f 2π-per., $f(x) = x(x^2-\pi^2)$, $x \in (-\pi, \pi]$

a) $\hat{f}(n) \overset{n\neq 0}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)e^{-inx}\,dx$

$\overset{3\times\text{ i.b.p.}}{=} 3 \cdot (-1)^n \frac{6}{i} \cdot \frac{1}{n^3}$ [see sol'ns]

$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)\,dx \overset{\text{odd}}{=} 0$

b) Obs 1: $f(x) \to 0$, $x \to \pm\pi$

$\Rightarrow f \in C_{per}(\mathbb{R}) \simeq C(\mathbb{S})$

Obs. 2: $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \leq 6 \sum \frac{1}{n^3} < \infty$.

$f \in C(\mathbb{S}) + \sum_{n\in\mathbb{Z}}|\hat{f}(n)| < \infty \overset{\text{Thm}}{\Longrightarrow}$ F-series of f conv. unif. to f:

$\| S_N(f) - f\|_\infty \underset{N\to\infty}{\longrightarrow} 0$

c) $\zeta(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\zeta(6) = \sum_{n=1}^{\infty}\frac{1}{n^6}$

Parseval: $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2 = \|f\|^2 = \|\hat{f}\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = 36 \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^6}$

$= 36 \cdot \left(\sum_{n=1}^{\infty} + \sum_{n=-\infty}^{-1}\right)\frac{1}{n^6} = 36 \cdot 2 \cdot \zeta(6)$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2(x^2-\pi^2)^2\,dx = \underset{\text{see sol}}{\ldots} = \frac{8\pi^6}{105}$

Hence $\zeta(6) = \frac{1}{36\cdot 2}\cdot\frac{8\pi^6}{105} = \frac{\pi^6}{945}$

From exercises:

$$\left[(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f\,\bar{g}\,dx \;\left(= \frac{1}{2\pi}(f,g)_2\right),\quad \|f\|^2 = \frac{1}{2\pi}\|f\|_2^2\right]$$

2.6:10 and 3.3:13 Obs: $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

$$f \in C^k(\$) \Rightarrow |\hat{f}(n)| \le \frac{C}{|n|^k},\; n \neq 0 \quad \text{and} \quad |\hat{f}(n)| = o\left(\frac{1}{|n|^k}\right),\; |n|\to\infty$$

Pf.: $\cancel{\left[\text{Obs: } |\hat{f}(n)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f| = \frac{1}{2\pi}\|f\|_1 \le \frac{1}{2\pi}2\pi\|f\|_\infty\right]}$.

$$f\in C^k(\$) \Rightarrow \widehat{f^{(k)}}(n) = \frac{1}{2\pi}\int f^{(k)}(x)e^{-inx}\,dx = (in)^k\hat{f}(n)$$

$$\overset{n\neq 0}{\Rightarrow} |\hat{f}(n)| \le \frac{1}{|n|^k}\cdot|\widehat{f^{(k)}}(n)| \overset{\text{chk}}{\le} \cancel{\frac{1}{|n|^k}\cdot\frac{1}{2\pi}\|f^{(k)}\|_1}$$

$$\overset{\text{chk}}{\le} \|f^{(k)}\|_\infty\cdot\frac{1}{|n|^k} \qquad \square$$

3.3:13

$$f\in C^k(\$) \Rightarrow |\hat{f}(n)| = o\left(\frac{1}{|n|^k}\right),\; n\to\infty$$

Pf.:

$$f\in C^k(\$) \Rightarrow f^{(k)} \text{ bnd} \Rightarrow f^{(k)} \in L^2(-\pi,\pi)$$

By Riemann-Lesbegue lem.

$$(in)^k\hat{f}(n) = \widehat{f^{(k)}}(n) \xrightarrow[n\to\infty]{f^{(k)}\in L^2} 0 \qquad \square$$

3.3:15 f R.int. on $\$$ (2π-per.)

(a) $$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx+in\frac{\pi}{n}}\cdot\underbrace{e^{-i\pi}}_{=-1}\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-in(x-\frac{\pi}{n})}\,dx$$

$$\overset{y=x-\frac{\pi}{n}}{=} -\frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} \underbrace{f\left(y+\frac{\pi}{n}\right)}_{2\pi\text{-per.}} e^{-iny}\,dy$$

$\int_0^L g = \int_k^{k+L} g$ if g L-per. **2)**

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(y+\frac{\pi}{n}\right) e^{-iny}\, dy$$

(*) Hence: $\hat f(n) = \frac12 \hat f(n) + \frac12\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n}\right) e^{-inx}dx\right) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x)-f\left(x+\frac{\pi}{n}\right)\right)e^{-inx}dx$

(b) f $\alpha\in(0,1]$ Hölder cont.: $|f(x+h)-f(x)| \le C|h|^\alpha$ $\forall\, x,h$

$$\Longrightarrow |\hat f(n)| \overset{(*)}{\le} \frac{1}{4\pi}\int_{-\pi}^{\pi} \overbrace{\left|f(x)-f\left(x+\frac{\pi}{n}\right)\right|}^{\le C|\frac{\pi}{n}|^\alpha} dx$$

$$\le \frac{1}{4\pi} C\left|\frac{\pi}{n}\right|^\alpha \cdot 2\pi = \frac{\frac12 \pi^\alpha C}{|n|^\alpha}$$

□

(c) (b) is optimal because

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}, \quad \alpha\in(0,1),$$

is well-def. ($\sum (2^{-\alpha})^k = \frac{1}{1-2^{-\alpha}} \Rightarrow$ abs. + unif. conv. $\forall x$),

and

(i) $\hat f(n) = \begin{cases} 2^{-k\alpha} = n^{-\alpha}, & n = 2^k \text{ for } k\in\mathbb{N} \\ 0, & \text{otherwise}\end{cases}$

(ii) $|f(x+h)-f(x)| \le C|h|^\alpha$ $\forall\, x,h$ [See sol'ns]

Pf. (ii): ($|h|<1$)

$$|f(x+h)-f(x)| \le \sum_{k=0}^{\infty} 2^{-k\alpha}\left|e^{i2^k(x+h)} - e^{i2^k x}\right|$$

$$\le \sum_{k:\,2^k\le\frac{1}{|h|}} 2^{-k\alpha}\underbrace{|e^{i2^k x}|}_{=1}\cdot\underbrace{|e^{i2^k h}-1|}_{\le |2^k h|} + \underbrace{\sum_{k:\,2^k>\frac{1}{|h|}} 2^{-k\alpha}\underbrace{\left(|e^{i2^k(x+h)}|+|e^{i2^k x}|\right)}_{\le 2}}$$

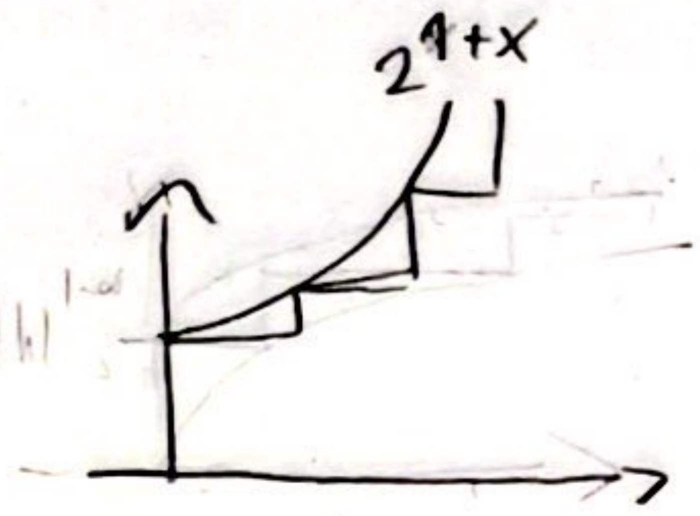
Hint. $|e^{ix}-1| = |(ix)\cdot(e^{ix})'_{x=\xi}| \le |x|$

$k=\bar k+m$

$$\le 2\cdot \underbrace{2^{-\bar k\alpha}}_{\le |h|^\alpha}\sum_{m=0}^{\infty} 2^{-m\alpha} \le 2|h|^\alpha\cdot\frac{1}{1-2^{-\alpha}}$$

$\bar k = \min\{k : 2^k > \frac{1}{|h|}\}$

$$\le \sum_{k:\, 2^k \le \frac{1}{|h|}} 2^{-k\alpha} |h| 2^k + C|h|^\alpha \le C|h|^\alpha$$



Better:

$$\underbrace{|h| \sum_{k:\, 2^k \le \frac{1}{|h|}} 2^{k(1-\alpha)}}$$

$$\le \int_0^{-\frac{\ln|h|}{\ln 2}} \underbrace{2^{(1+x)(1-\alpha)}}_{2^{1-\alpha} e^{x(1-\alpha)\ln 2}} dx \qquad \left[2^k = \frac{1}{|h|} \Rightarrow k \ln 2 = \underset{0}{\ln 1} - \ln|h| \right]$$

$$\le 2^{1-\alpha} \frac{1}{(1-\alpha)\ln 2} \underbrace{\left[e^{x(1-\alpha)\ln 2} \right]_0^{-\frac{\ln|h|}{\ln 2}}}_{= |h|^{-(1-\alpha)}}$$

$$|h|^\alpha \underbrace{\sum_{k:\, |h|2^k \le 1} (|h|2^k)^{1-\alpha}}$$

$$\le \sum_{k=0}^{\tilde{k}} (|h|2^k)^{1-\alpha}, \qquad |h| \le 2^{-\tilde{k}}$$

$$\tilde{k} = \max\{k : 2^k \le \tfrac{1}{|h|}\}$$

$$\le \sum_{k=0}^{\tilde{k}} (2^{1-\alpha})^{-(\tilde{k}-k)}$$

$$l = \tilde{k} - k: \quad = \sum_{l=0}^{\tilde{k}} (2^{1-\alpha})^{-l} \le \frac{1}{1 - 2^{\alpha - 1}}$$

3.3: 17

$f: \mathbb{R} \to \mathbb{R}$ 2π-per., bnd, monotone (nondecr.) on $[-\pi, \pi]$ $\Rightarrow$ $\hat{f}(n) \le \frac{C}{|n|}$, $n \ne 0$

Pf.:

1) $\widehat{\chi_{(a,b)}}(n) \overset{n\ne 0}{=} \frac{1}{2\pi} \int_a^b e^{-inx} dx = \frac{1}{2\pi} \frac{1}{in} \left(e^{-ina} - e^{-inb} \right) \left(= O\left(\frac{1}{|n|}\right) \right)$

2) $s_N(x) = \sum_{k=1}^N c_k \chi_{(a_k, a_{k+1})}(x) \Longrightarrow \widehat{s_N}(n) \overset{n \ne 0}{=} \sum_{k=1}^N c_k \widehat{\chi}_{(a_k, a_{k+1})}(n)$

3) Assume $-\pi = a_1 < a_2 < \dots < a_{N+1} = \pi$, $-M \le c_1 \le \dots \le c_N \le M$.

Then s_N bnd, mon., $|s_N| \le M$, and

$$2\pi i n\, \widehat{s_N}(n) \overset{n\ne 0}{=} \sum_{k=1}^N c_k \left(e^{-ina_k} - e^{-ina_{k+1}} \right)$$

$$= \sum_{l=0}^{N-1} c_{l+1} e^{-ina_{l+1}} - \sum_{k=1}^N c_k e^{-ina_{k+1}}$$

$$= c_1 e^{-ina_1} + \sum_{k=1}^{N-1} (c_{k+1} - c_k) e^{-ina_{k+1}} - c_N e^{-ina_{N+1}}$$

Hence

$$2\pi |n| \cdot |\hat{s}_N| \overset{\Delta\text{-ineq.}}{\le} |c_1| + \underbrace{\sum_{k=1}^{N-1} \overbrace{(c_{k+1} - c_k)}^{\ge 0}}_{\text{telescoping}} + |c_N|$$

$$= |c_1| + c_N - c_0 + |c_N| \le 2(|c_1| + |c_N|) \le 4M$$

$$|\hat{s}_N(n)| \le \frac{2M}{\pi |n|} \quad (\text{indep. of } N)$$

4) Assume f real-valued.

Let $a_k = -\pi + k \cdot \frac{2\pi}{N+1}$, $c_k = f(a_k)$, then

$$s_N(x) \le f(x) \le \sum_{k=1}^{N} c_{k+1} \chi_{(a_k, a_{k+1})}(x) =: \tilde{s}_N(x) \quad [c_{N+1} := c_N]$$

and,

$$0 \le \int_{-\pi}^{\pi} (f(x) - s_N(x))\,dx \le \int_{-\pi}^{\pi} (\tilde{s}_N(x) - s_N(x))\,dx = \sum_{k=1}^{N} (c_{k+1} - c_k) \frac{1}{N+1}$$

$$= \frac{c_{N+1} - c_1}{N+1} \le \frac{2M}{N+1}$$

$$\Longrightarrow \|f - s_N\|_1 \le \frac{2M}{N+1}$$

5) $|\hat{f}(n)| \le |\hat{s}_N(n)| + |\hat{f}(n) - \hat{s}_N(n)|$

$$\le \frac{2M}{|n|} + \int_{-\pi}^{\pi} |f - s_N| \cdot |e^{-inx}|\,dx$$

$$\le \frac{2M}{|n|} + \frac{2M}{N+1}$$

Send $N \to \infty$:

$$|\hat{f}(n)| \le \frac{2M}{|n|}, \quad n \neq 0$$

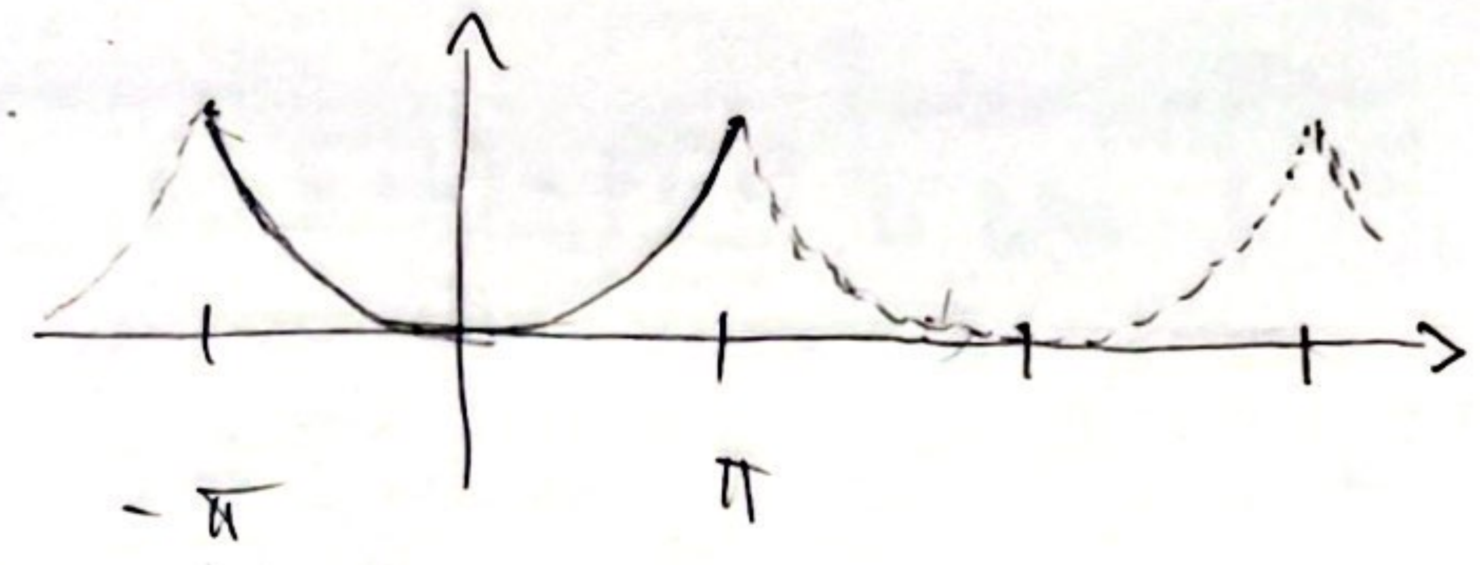
$\square$

<u>Rem</u>: Well known (i) step-f'ns s_N can approx R.int. f in $\|\cdot\|_1$

(ii) $\hat{\chi}_{(a,b)} = O(\frac{1}{n})$

Monotonicity $\Rightarrow$ $|\hat{s}_N(n)| \le \frac{C}{|n|}$, C indep of N !

Ex. 5 : P1 : $f(x) = x^2, \quad x \in [-\pi, \pi]$



a) $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$\hat{f}(n) \overset{n\neq 0}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \dots \overset{2\times \text{i.b.p.}}{=} 2\frac{(-1)^n}{n^2}$

$S_f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{inx}$

L^2-conv.:

b) $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{5}\pi^4 < \infty$

$\overset{f \text{ conti.}}{\Longrightarrow} f \in L^2(-\pi,\pi) \quad [\, f \in C + \text{bnd} \Rightarrow f \text{ R. int.} \,]$

$\overset{\text{Lectures}}{\Longrightarrow} \|\underbrace{S_{f,N}}_{\sum_{|n|\le N} \hat{f}(n)e^{inx}} - f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_{f,N}(x) - f(x)|^2\,dx \underset{N\to\infty}{\longrightarrow} 0$

Hence: S_f : conv. in $\|\cdot\|$ (L^2), $(S_{f,N} \xrightarrow{L^2} S_f)$, and

$\|S_f - f\| = 0 \quad (\Rightarrow f(x) = S_f(x) \text{ a.e. } x)$

c) (i) $f(x) = x^2$ in $(-\pi,\pi) \Rightarrow f \in C^\infty(-\pi,\pi)$

(ii) $\max_{x\in[-\pi,\pi]} |\underbrace{f'(x)}_{2x}| = 2\pi \Rightarrow |f(x+h) - f(x)| \overset{\text{Taylor}}{=} |f'(\xi)\cdot h| \le 2\pi |h|$

i.e. f Lipschitz

(iii) One-sided deriv's:

$D^+f(x) = D^-f(x) = f'(x), \quad x \in (-\pi,\pi)$

f 2π-per.

$$\Rightarrow D^{\pm}_{x} f(-\pi) = D^{\pm}_{x} f(\pi)$$

$$D^{+}_{x} f(-\pi) = \lim_{h\to 0^+} \frac{f(-\pi+h)-f(-\pi)}{h} \overset{\text{Taylor}}{=} \lim_{h\to 0^+} f'(\xi_h) = \underline{-2\pi}, \quad f'(\xi_h) = 2\xi_h, \quad \xi_h \in (-\pi, -\pi+h) \underset{h\to 0}{\longrightarrow} -\pi$$

$$D^{-}_{x} f(\pi) = \lim_{h\to 0} f'(\tilde{\xi}_h) = \underline{2\pi}, \quad f'(\tilde{\xi}_h) = 2\tilde{\xi}_h, \quad \tilde{\xi}_h \in (\pi-h, \pi) \underset{h\to 0}{\longrightarrow} \pi$$

(iv) $D^{+}_{x} f(\pi) \neq D^{-}_{x} f(\pi) = D^{-}_{x} f(-\pi) \Rightarrow f'(\pi)$ does not exist

$$\Rightarrow f \notin C^1(\mathbb{S}) \simeq C^1_{per}(\mathbb{R})$$

d) Pt.w. conv: f R.-int. (L^1), $f(x^{\pm})$ and $D^{\pm}_{x} f(x)$ ex. in every pt., f cont.

$$\overset{\text{Lectures}}{\Rightarrow} S_N(f)(x) \longrightarrow \frac{1}{2}(f(x^-)+f(x^+)) \underset{f \text{ cont.}}{=} f(x) \quad \forall x$$

e) Unif. conv.:
Result: $|f(x+h) - f(x)| \leq C|h|^{\alpha} \quad \forall x, h$.

If $\alpha \in (\frac{1}{2}, 1]$, then $S_N(f) \to f$ abs. + unif.

Here: $\alpha = 1$, so unif. conv. follows.

Ex. 5 P2:

Use Lem.: f R.-int. (L^1), $k \in \mathbb{N}$, $\beta > 1$, $|\hat{f}(n)| \leq \frac{C}{|n|^{k+\beta}}$, $n \neq 0$

$$\Rightarrow f \in C^k(\mathbb{S})$$

a) f R.-int., $|\hat{f}(n)| \leq \frac{C}{|n|^2} \underset{k=0, \beta=2}{\overset{\text{Lem.}}{\Rightarrow}} f \in C^0(\mathbb{S})$

b) — " —, $|\hat{f}(n)| \leq e^{-|n|} \Rightarrow |n|^{k+2} |\hat{f}(n)| \leq \sup_{m\in\mathbb{Z}} |m|^{k+2} e^{-|m|} < \infty \underset{k, \beta=2}{\overset{\text{Lem}}{\Rightarrow}} f \in C^{(k)}(\mathbb{S}) \quad \forall k \in \mathbb{N}$

÷ Ex. 3 P1:

$$S(x) := \sum_{n\in\mathbb{Z}\setminus\{0\}} \underbrace{\underbrace{\frac{1}{n}\sin\left(\frac{5}{n}\right)}_{c_n} e^{i2nx}}_{f_n(x)}$$

Obs: $|\sin x| \le |x|$

$$\overset{n\neq 0}{\Longrightarrow} |c_n| \le \frac{1}{|n|}\cdot\frac{5}{|n|} =: M_n$$

Hence

$$|f_n(x)| \le M_n \text{ and } \sum_{n\neq 0} M_n < \infty,$$

so Weierstrass M-test implies

$$S(x) = \lim_{N\to\infty} \underset{\displaystyle\sum_{0<|n|\le N} f_n(x)}{\underset{\shortparallel}{S_N(x)}} \quad \text{conv. abs. and unif.}$$

Since f_n cont., S unif. lim. of cont. f'ns and hence is cont.

Exam 2021 P3

a) $f(x) = (x^2+2x+2)\,e^{-\pi x^2}$: $\quad \hat{f}(\xi) = ?$

Obs: $e^{-\pi x^2} \in S(\mathbb{R}) \iff \|x^k (e^{-\pi x^2})^{(n)}\|_\infty < \infty \ \forall\, k,n$

$$\implies \|x^k f^{(n)}\|_\infty < \infty \ \forall\, k,n \iff f \in S(\mathbb{R})$$

Obs 2: $\widehat{e^{-\pi x^2}}(\xi) = e^{-\pi \xi^2} =: K(\xi)$, $\quad K'(\xi) = -2\pi\xi K(\xi)$

$$\mathcal{F}[-2\pi i x f](\xi) = \hat{f}\,'(\xi)$$

Hence

$$\hat{f} = \mathcal{F}[x^2 K] + 2\mathcal{F}[xK] + 2\mathcal{F}[K]$$

$$= \frac{1}{(-2\pi i)^2}\mathcal{F}[(-2\pi i x)^2 K] + \frac{2}{(-2\pi i)}\mathcal{F}[(-2\pi i x)K] + 2\hat{K}$$

$$= -\frac{1}{4\pi^2}\hat{K}'' - \frac{1}{\pi i}\underbrace{\hat{K}'}_{-2\pi\xi\hat{K}} + 2\hat{K}$$

$$= -\frac{1}{4\pi^2}(-2\pi\xi\hat{K})' - \frac{1}{\pi i}(-2\pi\xi\hat{K}) + 2\hat{K}$$

$$= -\frac{1}{4\pi^2}\Big(-2\pi\hat{K} - 2\pi\xi\underbrace{\hat{K}'}_{-2\pi\xi\hat{K}}\Big) + \Big(\frac{2}{i}\xi + 2\Big)\hat{K}$$

$\hat{K} = e^{-\pi\xi^2}$

$$= \Big(-\xi^2 - 2i\xi + \big(2 + \tfrac{1}{2\pi}\big)\Big)\, e^{-\pi\xi^2}$$

b) $g: \mathbb{R} \to \mathbb{R}$ def. by

$$e^{\pi t^2} \cdot g(t) = \int_{\mathbb{R}} (x^2 + 2x + 2)\, e^{-2\pi x(x-t)}\, dx \ , \ t \in \mathbb{R}.$$

$\hat{g} = ?$

Obs: $$g(t) = \int_{\mathbb{R}} (x^2 + 2x + 2) \underbrace{e^{-2\pi(x^2 - tx)} \cdot e^{-\pi t^2}}_{e^{-\pi x^2} \cdot e^{-\pi \underbrace{(x^2 - 2tx + t^2)}_{=(x-t)^2}}} dx$$

$$= \int_{\mathbb{R}} f(x) \underbrace{e^{-\pi(x-t)^2}}_{= K(x-t)} dx$$

$$= (f * K)(t)$$

$$\Longrightarrow \hat{g}(\xi) \underset{\substack{\uparrow \\ f, K \in S(\mathbb{R}) \subset C_m(\mathbb{R})}}{=} \hat{f} \cdot \hat{K}(\xi) = \underline{\underline{\hat{f}(\xi) \cdot e^{-\pi \xi^2}}}$$

Ex. 9 P1:

a) $f_1(x) = (1-x^2)\chi_{[-1,1]}(x)$

$$\hat{f}_1(\xi) = \int_{\mathbb{R}} f_1(x) e^{-2\pi i x \xi} dx$$

$$= \int_{-1}^{1} (1-x^2) e^{-2\pi i x \xi} dx \overset{2\times \text{i.b.p}}{=} \dots = \underbrace{-\frac{1}{(\pi\xi)^2}\cos(2\pi\xi) - i\frac{1}{2}\frac{1}{(\pi\xi)^3}\sin(2\pi\xi)}_{=: f_2(\xi)}$$

b) Alternative: $\hat{\chi}_{[-1,1]}(\xi) = \int_{-1}^{1} e^{-2\pi i x\xi} dx$

$$= \frac{1}{-2\pi i \xi}\left(e^{-2\pi i \xi} - e^{+2\pi i \xi}\right)$$

$$= \frac{\sin 2\pi\xi}{\pi\xi}$$

$$\hat{f}_1 = \hat{\chi}_{[-1,1]} - \frac{1}{(-2\pi i)^2}\mathcal{F}\left[(-2\pi i x)^2 \chi_{[-1,1]}(x)\right]$$

$$(*) \qquad = \underbrace{\hat{\chi}_{[-1,1]} + \frac{1}{4\pi^2}\hat{\chi}''}_{f_2(\xi)}, \quad \text{compute } \hat{\chi}''$$

b) $f_2(x)$, $\hat{f}_2 = ?$

Obs: $f_2 \overset{|x| \ll 1}{\sim} \frac{1}{x^2} \Rightarrow \int_{|x|<1} |f_2| = \infty \Rightarrow f_2 \notin C_m$ (or L^1 or L^2)!

But $$\int_{|x|\geq r} \hat{f}_2 \bar{\varphi} \overset{(*)}{=} \int_{|x|\geq r} \hat{\chi}_{[-1,1]} \bar{\varphi} + \frac{1}{4\pi^2}\underbrace{\int_{\mathbb{R}} \hat{\chi}'' \bar{\varphi}}_{\overset{2\times\text{i.b.p.}}{=} \int_{\mathbb{R}} \hat{\chi}\bar{\varphi}''} = \underbrace{\int \hat{\chi}_{[-1,1]}\left(\bar{\varphi} + \frac{1}{4\pi^2}\bar{\varphi}''\right)}_{=: F_2(\varphi)}$$

$\varphi \in S(\mathbb{R})$

$\varphi(0) = \varphi'(0) = 0$

$f_2 \sim F_2 \in S'(\mathbb{R})$, a Schwarz distribution!

Outside current curriculum, but $\hat{f}_2(x)\,\hat{f}_1 \Rightarrow \hat{f}_2(x) = f_1(-x)$

$$f_2 = \hat{f}_1 \Rightarrow \hat{f}_2(x) = \hat{\hat{f}}_1(x) = \check{\hat{f}}_1(-x) = f_1(-x) \overset{f_1 \text{ even}}{=} f_1(x)$$

$\mathcal{F}^{-1}[f](\xi) = \mathcal{F}[f(-x)](\xi)$ $\mathcal{F}$-inv.
$= \mathcal{F}[f](-\xi)$ in $\underline{S'(\mathbb{R})}$

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Hence $\hat{f}_2 = f_1$ in $\mathbb{R}$ (since f_1 is a f'n).

Ref-to Exam 2021 (similar)

c) $f_3(x) = (1-x^2)\underbrace{e^{-\pi x^2}}_{=K(x)}$, $\hat{K} = K$, $K'(\xi) = -2\pi\xi K(\xi)$

$$\hat{f}_3(\xi) = \hat{K}(\xi) - \frac{1}{(-2\pi i)^2}\underbrace{\mathcal{F}[(-2\pi i x)^2 K(x)](\xi)}_{=\hat{K}''(\xi)}$$

$$\left[= \hat{K}(\xi) + \frac{1}{4\pi^2}\hat{K}''(\xi)\right. \left[= K(\xi) + \frac{1}{4\pi^2}\left(-2\pi\xi K(\xi)\right)'\right]$$

$$= \cdots = \left(1 - \frac{1}{2\pi} + \xi^2\right)K(\xi)$$

d) $f_4(x) = x\, e^{-\pi a^2 x^2} = x \cdot \hat{K}(ax)$

Obs: $\mathcal{F}[f(ax)](\xi) = a^{-\frac{1}{2}}\hat{f}(a^{-\frac{1}{2}}\xi)$

$\Rightarrow \mathcal{F}[K(ax)](\xi) = a^{-\frac{1}{2}}K(a^{-\frac{1}{2}}\xi) =: K_a(\xi)$, $K_a' = -2\pi a^{-\frac{1}{2}}\xi K_a(\xi)$

$$\hat{f}_4(\xi) = \frac{1}{-2\pi i}\mathcal{F}[(-2\pi i x)K(ax)](\xi) = \frac{1}{-2\pi i}K_a'(\xi)$$

$$= \frac{1}{-2\pi i}\left(-2\pi a^{-\frac{1}{2}}\xi\right)K_a(\xi) = \frac{\xi}{\pi i a^{\frac{1}{2}}}K_a(\xi) = \underline{\underline{\frac{1}{\pi i}\left(\frac{\xi}{a}\right)e^{-\pi\frac{\xi^2}{a}}}}$$

Ex. 9 P2:

(i) $f(x) = \frac{\sin(2\pi x)}{x} \chi_{[0,\infty)}(x)$

$$\left.\begin{array}{l} \text{Obs!: } f(0^+) = 2\pi \lim\limits_{x\to 0} \frac{\sin 2\pi x}{2\pi x} = \underline{2\pi} \\ f(0^-) = 0 \\ f \in C(\mathbb{R}\setminus\{0\}) \\ |f| \to 0 \,,\; |x| \to \infty \\ \|f\|_\infty \le f(0^+) = 2\pi \end{array}\right\} \begin{array}{c} f \in PC_0(\mathbb{R}) \\ \text{p.w. cont.} \end{array}$$

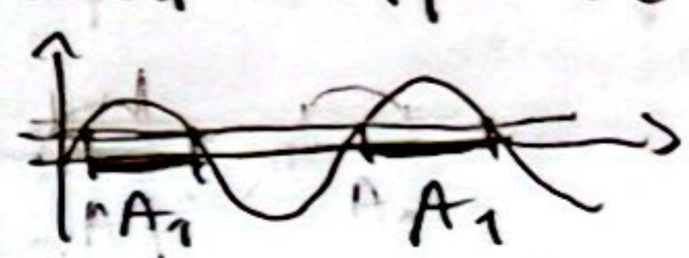
$\underline{f \notin PC_m(\mathbb{R})}$: $\|x^{1+\varepsilon} f(x)\|_\infty = \|x^\varepsilon \sin 2\pi x\|_\infty > \left(\frac{1}{4}+k\right)^\varepsilon \sin\left(\frac{\pi}{2} + k2\pi\right) = \left(\frac{1}{4}+k\right)^\varepsilon \;\forall k > 0$

$x = \frac{1}{4} + k$

moderate decrease

$\Rightarrow \|x^{1+\varepsilon} f(x)\|_\infty = \infty$, not moderate decrease !

$\underline{f \notin L^1(\mathbb{R})}$: $A_1 := \{x \ge 1 : |\sin 2\pi x| \ge \frac{1}{2}\}$; $[\frac{\pi}{6}, \pi - \frac{\pi}{6}] + k\cdot 2\pi \subset A_1 \;\forall k \in \mathbb{N}$



$$\int_{\mathbb{R}} |f|\,dx = \lim_{R\to\infty} \int_{|x|\le R} |f|\,dx \ge \lim_{R\to\infty} \int_{|x|\le R \cap A_1} \frac{\frac{1}{2}}{x}\,dx$$

$$\ge \lim_{R\to\infty} \frac{1}{2} \sum_{k\in\mathbb{N}:\, \frac{5}{6}\pi + k2\pi \le R} \underbrace{\int_{\frac{\pi}{6}+k\cdot 2\pi}^{\frac{5\pi}{6}+k\cdot 2\pi} \frac{1}{x}\,dx}$$

$$\ge \frac{1}{\frac{5\pi}{6}+k\cdot 2\pi} \ge \frac{1}{(2k+1)\pi}$$

$$\ge \frac{1}{2} \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)\pi} \cdot \underbrace{\int_{\frac{\pi}{6}+k\cdot 2\pi}^{\frac{5}{6}\pi+k\cdot 2\pi} dx}_{= \frac{4}{6}\pi = \frac{2}{3}\pi} = \frac{1}{3} \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)}$$

$$= \frac{1}{3} \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)} \ge \frac{1}{3} \int_1^\infty \frac{1}{x}\,dx = \infty$$

$\underline{f^2 \in PC_m(\mathbb{R})}$: $|f(x)| \le 2\pi$ and $|f(x)| \le \frac{1}{|x|}$

$|\sin| \le 1$ $\quad |\sin| \le 1$ $\quad |f| \le 2\pi$

$$\Rightarrow \|x^{1+\varepsilon} f(x)^2\|_\infty \le \|x^{1+\varepsilon} f^2 \chi_{[0,1]}\|_\infty + \|x^{1+\varepsilon} f^2 \chi_{[1,\infty)}\|_\infty$$

$$\le 1^{1+\varepsilon}(2\pi)^2 + \|x^{1+\varepsilon-2}\chi_{[1,\infty)}\|_\infty < \infty$$

$f \in L^2(\mathbb{R})$: $f \in PC(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f|^2 dx = \lim_{R\to\infty} \int_{-R}^{R} |f|^2 dx = \underbrace{\int_{-1}^{1} |f|^2}_{\le (2\pi)^2} + \lim_{R\to\infty}\left(\int_{-R}^{-1} + \int_{1}^{R}\right) \underbrace{|f|^2}_{\le \frac{1}{|x|^2}}$$

$$\le (2\pi)^2 \cdot 2 + \lim_{R\to\infty}\left(\int_{-R}^{-1} + \int_{1}^{R}\right) \frac{1}{|x|^2} dx$$

$$= 8\pi^2 + \lim_{R\to\infty} 2\cdot\left(\frac{1}{1} - \frac{1}{R}\right) = 8\pi^2 + 2 < \infty$$

$[f \in PC_m \overset{!}{\Rightarrow} f \in L^2]$

Since $f \in L^2(\mathbb{R})$,

$g(x) := \mathcal{F}^{-1}[\hat{f}](x) = f(x)$ in L^2 $(\Leftrightarrow \|g - f\|_2 = 0)$ and a.e. almost everywhere.

Since $f \notin L^1(\mathbb{R})$ (or $C_m(\mathbb{R})$), we cannot say anything about pt. wise conv. $\underbrace{\text{from results in this course.}}_{\text{and } g(0),\, g(1)}$

(ii) $f(x) = \frac{1}{x}\chi_{[1,\infty)}(x)$ $\quad$ ($f \in PC$, bnd)

$\int |f| = \int_1^\infty \frac{1}{x} = \infty \Rightarrow f \notin L^1$

$\int |f|^2 = \int_1^\infty \frac{1}{x^2} < \infty \Rightarrow f \in L^2$

Hence $g(x) = \mathcal{F}^{-1}[f](x)$ in L^2 and a.e.,

no info about pt. conv. and $g(0)$, $g(1)$

3)

(iii) $f(x) = \frac{\sin 2\pi x}{x^2} \chi_{[1,\infty)}(x)$ (PC, bad)

$\int |f| \leq \int_1^\infty \frac{1}{x^2} < \infty \implies f \in L^1(\mathbb{R})$

$\int |f|^2 \leq \int_1^\infty \frac{1}{x^4} < \infty \implies f \in L^2(\mathbb{R})$

Chk.: $f \in C(\mathbb{R} \setminus \{1\})$

$$f(1^+) = \frac{\sin 2\pi}{1^2} = 0, \quad f(1^-) = 0$$

$$\implies f \in C(\mathbb{R}) \quad [\, f \in C_m(\mathbb{R})\,!\,]$$

$$D^+ f(1) = \lim_{x \to 1^+} f'(x) = \lim_{x \to 1} \left(\frac{2\pi \cos 2\pi x}{x^2} - 2 \overset{0}{\cancel{\frac{\sin 2\pi x}{x^3}}} \right) = 2\pi$$

$$D^- f(1) = 0$$

$$D^\pm f(x) = f'(x) \text{ for } x \neq 1$$

Hence by Thm. 3 lecture 3.3.2025 (Week 10),

$$g(x) = \mathcal{F}^{-1}[\hat{f}](x) = \tfrac{1}{2}\left(f(x^+) + f(x^-)\right) \overset{f \text{ cont.}}{=} f(x), \quad x \in \mathbb{R}$$

In particular, $g(0) = f(0) = 0$

$$g(1) = f(1) = 0$$

Heisenberg opg?